University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. Calculators are not needed, but you may use one if you wish. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points) A parallel-strip transmission line consists of two perfectly conducting long flat thin metal strips of width D and vacuum separation $d \ll D$ extending a long distance in the $\hat{\mathbf{z}}$ direction. Take $\hat{\mathbf{x}}$ to be normal to the strips, and define $x = y = 0$ at the midpoint of the gap. Consider the propagation in the gap of electromagnetic waves of angular frequency ω in the TEM mode ($E_z = B_z = 0$). Take

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\big(\tilde{\mathbf{E}}(x, y)e^{i(\kappa z-\omega t)}\big)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}\big(\tilde{\mathbf{B}}(x, y)e^{i(\kappa z-\omega t)}\big) ,$$

where $\kappa \equiv \omega/c$. Then, in vacuum, the relevant Maxwell equations reduce to

$$\begin{aligned} 0 &= \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} \\ 0 &= \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \\ c\tilde{B}_y &= \tilde{E}_x \\ -c\tilde{B}_x &= \tilde{E}_y \ . \end{aligned}$$

(**a**) (10 points) Show that $\tilde{\mathbf{E}}$ can be written as

$$\tilde{\mathbf{E}}(\mathbf{x}, \mathbf{y}) = -\nabla_t \tilde{\Phi}(x, y) ,$$

where

$$\nabla_t^2 \tilde{\Phi} = 0$$

and

$$\nabla_t \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} \ .$$

If you don't manage to show this, you nevertheless should assume this result in the later parts.

Solution:

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = 0 \\ \Rightarrow \tilde{\mathbf{E}} &= -\nabla \tilde{\Phi} = -\nabla_t \tilde{\Phi} \\ 0 &= \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} \\ &= \nabla_t \cdot \tilde{\mathbf{E}} \\ &= \nabla_t \cdot (-\nabla_t \tilde{\Phi}) \\ &= -\nabla_t^2 \tilde{\Phi} \ . \end{aligned}$$

(**b**) (10 points) Assume that

$$\tilde{\Phi} = +\Phi_0/2$$

on the top plate, and

$$\tilde{\Phi} = -\Phi_0/2$$

on the bottom plate, where Φ_0 is a real constant. Neglecting the small region near the edges, calculate the *real physical* fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the gap between the plates, in terms of Φ_0.

Solution:
The solution $\tilde{\Phi}$ to Laplace's equation is unique, given the boundary conditions:

$$\tilde{\Phi} = \Phi_0 \frac{x}{d} \ .$$

Therefore

$$\begin{aligned} \tilde{\mathbf{E}} &= -\hat{\mathbf{x}}\frac{\Phi_0}{d} \\ \mathbf{E} &= -\hat{\mathbf{x}}\frac{\Phi_0}{d}\cos(\kappa z - \omega t) \\ c\mathbf{B} &= -\hat{\mathbf{y}}\frac{\Phi_0}{d}\cos(\kappa z - \omega t) \ . \end{aligned}$$

(**c**) (15 points) Find the characteristic impedance Z of this transmission line. Evaluate Z in ohms for the case $D = 100d$.
[*Hint*: One way to do this is to take Z to be the ratio of Φ_0 to the maximum current flowing on the inner surface of either plate. Assume that this current is distributed uniformly in y. Another way is to take Z to be $\sqrt{L/C}$, where L and C are the inductance and capacitance per unit length of the transmission line, and $\sqrt{1/LC}$ is the phase velocity of the wave.]
Solution:
Method 1:
There are no time-varying fields within the perfect conductors. Therefore, across the inner boundary of either conductor, from Ampère's law, ignoring directions and signs,

$$\begin{aligned}\Delta B &= \mu_0 K \\ B_{\text{max}} &= \mu_0 K_{\text{max}} \ ,\end{aligned}$$

where K_{max} is the maximum surface current density (amperes/m). The impedance is

$$\begin{aligned}Z &= \frac{\Phi_0}{K_{\text{max}} D} \\ &= \frac{\mu_0 \Phi_0}{B_{\text{max}} D} \\ &= \frac{c\mu_0 \Phi_0}{E_{\text{max}} D} \\ &= \frac{c\mu_0 \Phi_0 d}{\Phi_0 D} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d}{D} \\ &= 3.77\ \Omega \ .\end{aligned}$$

Method 2:

$$\begin{aligned}v_{\text{ph}} &= \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{LC}} \\ L &= \frac{\epsilon_0 \mu_0}{C} \\ C &= \frac{\epsilon_0 D}{d} \\ Z &= \sqrt{\frac{L}{C}} = \sqrt{\frac{\epsilon_0 \mu_0}{C^2}} \\ &= \sqrt{\frac{\epsilon_0 \mu_0 d^2}{\epsilon_0^2 D^2}} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d}{D} \\ &= 3.77\ \Omega \ .\end{aligned}$$

Problem 2. (40 points) "Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays isotropically to an (anti)muon (μ^+) and a neutrino (ν) via

$$\pi^+ \to \mu^+ + \nu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have 3/4 of the rest mass of a pion; neglect the neutrino mass.
(**a**) (15 points) Show that the surface muons travel at a speed which is a fraction $\beta_0 = 0.28$ of the speed of light.
Solution:
Let μ, π, and ν be the four-momenta of the muon, pion, and neutrino, respectively, with units such that $c = 1$. Enforcing energy-

momentum conservation,

$$\begin{aligned}\pi &= \mu + \nu \\ \nu &= \pi - \mu \\ \nu \cdot \nu &= (\pi - \mu) \cdot (\pi - \mu) \\ 0 &= \pi \cdot \pi + \mu \cdot \mu - 2\pi \cdot \mu \\ 0 &= m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \\ E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \ .\end{aligned}$$

Similarly, permuting the same equation, and using $E_\nu = p_\nu = p_\mu$,

$$\begin{aligned}\mu &= \pi - \nu \\ \mu \cdot \mu &= (\pi - \nu) \cdot (\pi - \nu) \\ m_\mu^2 &= \pi \cdot \pi + \nu \cdot \nu - 2\pi \cdot \nu \\ &= m_\pi^2 + 0 - 2m_\pi E_\nu \\ &= m_\pi^2 - 2m_\pi p_\mu \\ p_\mu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \ .\end{aligned}$$

Taking the ratio of these two results

$$\begin{aligned}\beta_0 &= \frac{p_\mu}{E_\mu} \\ &= \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \\ &= \frac{\frac{16}{9} - 1}{\frac{16}{9} + 1} = \frac{7}{25} = 0.28 \ .\end{aligned}$$

(**b**) (15 points) A good method for capturing and transporting surface muons is to place the muon production target on the axis of a solenoidal magnet with uniform field B; this axis defines the beam direction. Muons (of charge e and rest mass m) that are emitted close to the axial direction are captured and transported by the solenoid. In terms of β_0 and other constants, over what path length L does a surface muon travel before it returns to the solenoid axis?

Solution:

The motion is helical with angular frequency equal to the (relativistic) cyclotron frequency. Working in the lab,

$$\begin{aligned}\Omega_{\text{cyclotron}} &= \frac{eB}{\gamma m} \\ &= \frac{eB\sqrt{1-\beta_0^2}}{m} \\ T &= \frac{2\pi}{\Omega_{\text{cyclotron}}} \\ &= \frac{2\pi m}{eB\sqrt{1-\beta_0^2}} \\ L &= \beta_0 c T \\ &= \frac{2\pi m \beta_0 c}{eB\sqrt{1-\beta_0^2}} \ .\end{aligned}$$

(**c**) (10 points) If a muon's mean proper lifetime is τ, what fraction of the muons will decay before they return to the solenoid axis? (If you are concerned that you didn't get part (**a**) or (**b**) quite right, you may leave your answer in terms of β_0 and L.)

Solution:

In the lab, the time interval before the muon returns to the solenoid axis is $T = L/(\beta_0 c)$ (above). In the proper (rest) frame of the muon, the same interval is $T' = T/\gamma_0$. If the mean life is τ, the survival probability at time T' is $\exp(-T'/\tau)$. Therefore the fraction F of muons that fail to survive before returning to the solenoid axis is

$$\begin{aligned}F &= 1 - \exp(-T'/\tau) \\ &= 1 - \exp\left(-T/(\gamma_0\tau)\right) \\ &= 1 - \exp\left(-\frac{L}{c\tau}\frac{\sqrt{1-\beta_0^2}}{\beta_0}\right) \ .\end{aligned}$$

The above is an acceptable solution. Expressed in terms of the answer to (**b**), it is

$$F = 1 - \exp\left(-\frac{2\pi m}{eB\tau}\right) \ ,$$

independent of β_0.

Problem 3. (25 points) Consider the interaction of an electron of charge $-e$ and mass m with an (externally produced) electromagnetic field described by the four-potential A^μ. The interaction Lagrangian L_{int} in this case is

$$L_{\text{int}} = -\frac{e}{\gamma m} p_\mu A^\mu \ ,$$

where p^μ is the particle's four-momentum. Consider the canonical momentum

$$P^\mu \equiv p^\mu - eA^\mu \ .$$

If one applies the Euler-Lagrange equations to L_{int}, one discovers that if all four components of A^μ are independent of any spatial coordinate x^i, then P^i, the i^{th} component of P^μ, is *conserved*.

While these facts may seem like theoretical niceties, they can be of practical use. Consider a capacitor whose parallel plates lie in the xy plane. The inside of the bottom plate is at $z = 0$ and the inside of the top plate is at $z = d$. The bottom plate is grounded, and a positive voltage V_0 is applied to the top plate. The whole setup is bathed in a uniform magnetic field

$$\mathbf{B} = \hat{\mathbf{y}} B_0 \ ,$$

which can be derived from a vector potential

$$\mathbf{A} = \hat{\mathbf{x}} B_0 z \ .$$

An electron is emitted from the bottom plate in the z direction with negligible velocity. It is accelerated in the z direction toward the top plate by the electric field in the gap; however, as the electron gains velocity, the Lorentz force from the magnetic field bends it toward the x direction. The resulting motion is complicated.

(**a**) (15 points) Show that the x component of the electron's momentum varies only as a function of its altitude z, and find the dependence.

Solution:

The components of A^μ are

$$\begin{aligned} A^0 &= \frac{V}{c} = \frac{V_0 z}{cd} \\ \mathbf{A} &= \hat{\mathbf{x}} B_0 z \ . \end{aligned}$$

Each component of A^μ is independent of both x and y. Therefore, both the x and y components of P^μ are conserved. Since $\mathbf{A}$ has no y component, conservation of the y component of P^μ merely confirms that the electron moves in the xz plane, which we could have deduced from the Lorentz force law. In the x direction,

$$\begin{aligned} p_x - eA_x &= \big(p_x(z=0) - eA_x(z=0)\big) \\ &= 0 - 0 \\ p_x &= eA_x \\ &= eB_0 z \ . \end{aligned}$$

(**b**) (10 points) For simplicity assuming that the electron is nonrelativistic, and taking B_0 to be fixed, find the minimum value of the applied voltage V_0 such that the electron makes it all the way up to the top plate.

[The above describes an oversimplified version of the *static magnetron tube*, which generated the radar signals that won the Battle of Britain.]

Solution:

If the electron barely grazes the top plate, it will be travelling parallel to it, or entirely in the x direction. Since the magnetic field does no work, the electron's kinetic energy at that point will be equal to its loss of potential energy eV_0. Using the result from part (**a**),

$$\begin{aligned} eV_0 &= \frac{p_x^2}{2m} \\ V_0 &= \frac{e^2 B_0^2 d^2}{2me} \\ &= \frac{eB_0^2 d^2}{2m} \ . \end{aligned}$$